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An algebraic study on the A_{N-1}- and B_N-Calogero models with bosonic, fermionic and distinguishable particles

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Abstract
Through an algebraic method using the Dunkl–Cherednik operators, the multivariable Hermite and Laguerre polynomials associated with the A_{N-1}- and B_N-Calogero models with bosonic, fermionic and distinguishable particles are investigated. The Rodrigues formulae of column type that algebraically generate the monic non-symmetric multivariable Hermite and Laguerre polynomials corresponding to the distinguishable case are presented. Symmetric and anti-symmetric polynomials that respectively give the eigenstates for bosonic and fermionic particles are also presented by the symmetrization and anti-symmetrization of the non-symmetric ones. The norms of all the eigenstates for all cases are algebraically calculated in a unified way.

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1. Introduction

In the early 1970s, one-dimensional quantum integrable systems with inverse-square long-range interactions appeared as a new class of nontrivial solvable models, now generally called the Calogero–Moser–Sutherland (CMS) models [4, 27, 37, 38, 50] in memory of the pioneers. Among the CMS models, the Calogero and the Sutherland models [4, 37, 38] are considered to be the most typical models. The models describe many-body systems confined by the external harmonic well or the periodic boundary condition, which are typical of the models in condensed matter physics. In particular, the Sutherland model attracted many researchers because, as early as the 1990s, its orthogonal basis was known to be the Jack polynomial [13, 23, 36]. The theory of the Jack polynomial enabled a calculation of the exact correlation functions of the Sutherland model [11, 15, 42] and a related model in condensed matter physics [16].

The quantum integrabilites of the two models, in a sense that they have as many commutative conserved operators as the number of degrees of freedom of the system, are explicitly shown by the Dunkl–Cherednik operator formulations of a common structure to the two models [5, 9, 34]. The formulations are extended and generalized from the point of view of the affine root systems so as to cover a wide class of the CMS models and to clarify relationships with other integrable systems [18, 19]. The celebrated symmetric Jack polynomials [13, 23, 36] are the simultaneous eigenfunctions of conserved operators constructed from the Cherednik operators of the Sutherland model. However, only a small amount of information was known about the symmetric simultaneous eigenfunctions of the conserved operators of the Calogero model which are made from Cherednik operators [44]. Motivated by the Rodrigues formula for the symmetric Jack polynomial that was found by Lapointe and Vinet [20, 21], we presented the Rodrigues formula for the Hi-Jack symmetric (multivariable Hermite) polynomial [22] and identified it as the simultaneous eigenfunction of the conserved operators of the Calogero model [45–47]. The multivariable Hermite polynomial is a one-parameter deformation of the symmetric Jack polynomial. They share many common properties, which reflect the same algebraic structure of the corresponding Dunkl–Cherednik operators. Moreover, the multivariable Laguerre polynomials as well as the above multivariable Hermite polynomials are investigated [1, 14, 49].

To study the Calogero and Sutherland models including spin variables, we need the non-symmetric simultaneous eigenvectors of the Cherednik operators as the orthogonal basis of the orbital part of the eigenstate [15, 40, 41]. Such a non-symmetric simultaneous eigenfunction of the conserved operators of the Sutherland model is known to be the non-symmetric Jack polynomial whose properties have been extensively studied in a mathematical context [17, 32, 33, 35, 39]. On the other hand, the simultaneous eigenfunction of the Calogero model is identified as the non-symmetric multivariable Hermite polynomial that is a one-parameter deformation of the non-symmetric Jack polynomial [2]. Some of the results for the non-symmetric Jack polynomials were translated to the theory of the non-symmetric multivariable Hermite and Laguerre polynomials [3, 28, 43, 48]. As is similar to the symmetric polynomial case, however, less properties are clarified on the non-symmetric multivariable Hermite and Laguerre polynomials than those of the non-symmetric Jack polynomials.

Recently, we investigated non-symmetric Jack and Macdonald polynomials and their symmetrization and anti-symmetrization by an algebraic formulation employing the Dunkl–Cherednik operators together with the theory of root systems [29, 30]. Through the method, algebraic constructions and evaluations of square norms for non-symmetric, symmetric and anti-symmetric multivariable polynomials can be treated in a unified way. In this paper, we shall extend and apply the above method to the multivariable Hermite and Laguerre polynomials. We shall present algebraic constructions of the non-symmetric polynomials, symmetrizations and anti-symmetrizations, and evaluate the square norms of the Hermite and Laguerre cases, which were not clarified in our previous works [28, 43, 45–48].

The outline of the paper is as follows. In section 2, we give a brief summary on the Dunkl–Cherednik operator formulation for the Calogero models. In section 3, the non-symmetric multivariable Hermite and Laguerre polynomials are introduced as the joint eigenvectors of the Cherednik operators. We also introduce a notation based on the A_{N-1}-root system associated with the finite-dimensional simple Lie algebra. In section 4, the algebraic constructions of the non-symmetric multivariable Hermite and Laguerre polynomials are presented. Square norms of the polynomials are calculated in an algebraic manner. In section 5, we construct symmetric and anti-symmetric polynomials and compute their square norms. The final section is devoted to a summary. A proof of a lemma is presented in the appendix.

2. Dunkl–Cherednik operators and the Calogero models

We give a brief summary on the Dunkl–Cherednik operator formulation of the A_{N-1}- and B_N-Calogero models [31], which were so named as a consequence of the fact that the corresponding root systems appear in their interaction terms. Actually, the B_N-Calogero model is the C_N-Calogero model at the same time, and reduces to the D_N-Calogero model by fixing the parameter $b = 0$. Thus the two models we shall study cover all the Calogero models associated with root systems of classical simple Lie algebras in the above-mentioned sense. The Hamiltonians of the Calogero models with distinguishable particles [2, 34, 51] are expressed as

$$\hat{\mathcal{H}}^{(A)} = \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2\right) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\frac{a^2 - aK_{jk}}{(x_j - x_k)^2} \tag{2.1a}$$

$$\hat{\mathcal{H}}^{(B)} = \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2 + \frac{b^2 - bt_j}{x_j^2}\right) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\left(\frac{a^2 - aK_{jk}}{(x_j - x_k)^2} + \frac{a^2 - at_jt_kK_{jk}}{(x_j + x_k)^2}\right) \tag{2.1b}$$

where the coordinate exchange operator K_{jk} and the reflection operator t_j are defined as

$$(K_{jk}f)(\ldots, x_j, \ldots, x_k, \ldots) = f(\ldots, x_k, \ldots, x_j, \ldots)$$
$$(t_j f)(\ldots, x_j, \ldots) = f(\ldots, -x_j, \ldots) \qquad j, k \in \{1, 2, \ldots, N\}$$

and we assume that the coupling parameters $a, b \in \mathbb{R}_{\geqslant 0}$. In general, the eigenstates of the above Calogero Hamiltonians (2.1) are non-symmetric with respect to exchanges of particle indices. This is why we have called them the models with distinguishable particles. The eigenfunctions of the Calogero models are expressed as the products of inhomogeneous non-symmetric multivariable polynomials, namely the non-symmetric multivariable Hermite and Laguerre polynomials [2, 10, 28, 48], and the reference states. To study the polynomial part of such eigenfunctions, we introduce transformed Hamiltonians whose eigenvectors are polynomials:

$$\mathcal{H}^{(A,B)} := \left(\phi_{\mathrm{g}}^{(A,B)}(x)\right)^{-1}_{\circ}\left(\hat{\mathcal{H}}^{(A,B)} - E_{\mathrm{g}}^{(A,B)}\right) \circ \phi_{\mathrm{g}}^{(A,B)}(x) \tag{2.2}$$

where

$$\begin{aligned}\phi_{\mathrm{g}}^{(A)}(x) &= \prod_{1\leqslant j<k\leqslant N}|x_j - x_k|^a \exp\left(-\tfrac{1}{2}\omega\sum_{m=1}^{N}x_m^2\right)\\ E_{\mathrm{g}}^{(A)} &= \tfrac{1}{2}\omega N\big(Na + (1-a)\big)\end{aligned} \tag{2.3a}$$

$$\begin{aligned}\phi_{\mathrm{g}}^{(B)}(x) &= \prod_{1\leqslant j<k\leqslant N}|x_j^2 - x_k^2|^a \prod_{l=1}^{N}|x_l|^b \exp\left(-\tfrac{1}{2}\omega\sum_{m=1}^{N}x_m^2\right)\\ E_{\mathrm{g}}^{(B)} &= \tfrac{1}{2}\omega N\big(2Na + (1-2a) + 2b\big).\end{aligned} \tag{2.3b}$$

The above reference states and their eigenvalues are known to be the ground states and the ground state energies for the A_{N-1}- and the B_N-Calogero models with distinguishable and bosonic particles. In the following, we call (2.2) instead of (2.1) the Calogero Hamiltonians.

Let $\mathbb{C}[x]$ be the polynomial ring with N variables over $\mathbb{C}$. We deal with the eigenfunctions for the original Calogero Hamiltonians $\hat{\mathcal{H}}^{(A,B)}$ in the spaces $\mathbb{C}[x]\phi_{\mathrm{g}}^{(A,B)} = \{f(x)\phi_{\mathrm{g}}^{(A,B)}(x) | f \in \mathbb{C}[x]\}$ with the following canonical inner products:

$$(\varphi, \psi) := \int_{-\infty}^{\infty}\prod_{j=1}^{N}\mathrm{d}x_j\,\overline{\varphi(x)}\psi(x) \qquad \text{for} \quad \varphi, \psi \in \mathbb{C}[x]\phi_{\mathrm{g}}^{(A,B)}$$

where $\overline{\varphi(x)}$ denotes the complex conjugate of $\varphi(x)$. On the other hand, the transformed Hamiltonians (2.2) are Hermitian with respect to the inner product on $\mathbb{C}[x]$:

$$\langle f, g\rangle_{(A,B)} := \int_{-\infty}^{\infty} \prod_{j=1}^{N} \mathrm{d}x_j \, |\phi_{\mathrm{g}}^{(A,B)}(x)|^2 \overline{f(x)} g(x) \qquad \text{for} \quad f, g \in \mathbb{C}[x] \tag{2.4}$$

which are induced from $(\cdot, \cdot)$ and the transformation (2.2). Thus the reference states (2.3) correspond to the weight functions in the above inner products $\langle \cdot, \cdot \rangle_{(A,B)}$. The commuting conserved operators for the Calogero Hamiltonians are known to be the Cherednik operators. To show this, we need to introduce the Dunkl operators $\nabla_j^{(A,B)} \in \mathrm{End}(\mathbb{C}[x])$ [9],

$$\nabla_j^{(A)} := \frac{\partial}{\partial x_j} + a \sum_{\substack{k=1\\k\neq j}}^{N} \frac{1}{x_j - x_k}(1 - K_{jk})$$

$$\nabla_j^{(B)} := \frac{\partial}{\partial x_j} + a \sum_{\substack{k=1\\k\neq j}}^{N} \left(\frac{1}{x_j - x_k}(1 - K_{jk}) + \frac{1}{x_j + x_k}(1 - t_j t_k K_{jk}) \right) + \frac{b}{x_j}(1 - t_j)$$

and the creation-like and annihilation-like operators $\alpha_l^{(A,B)\dagger}, \alpha_l^{(A,B)} \in \mathrm{End}(\mathbb{C}[x])$ for the Calogero models,

$$\alpha_l^{(A,B)\dagger} := x_l - \frac{1}{2\omega}\nabla_l^{(A,B)} \qquad \alpha_l^{(A,B)} = \frac{1}{2\omega}\nabla_l^{(A,B)}$$

where the superscript † on any operator denotes its Hermitian conjugate with respect to the inner product (2.4). From these operators, two sets of Hermitian and commutative differential operators $d_j^{(A,B)} \in \mathrm{End}(\mathbb{C}[x])$, $[d_j^{(A,B)}, d_k^{(A,B)}] = 0$, [3, 14] are constructed by

$$d_j^{(A)} := 2\omega \alpha_j^{(A)\dagger} \alpha_j^{(A)} + a \sum_{k=j+1}^{N} K_{jk}$$

$$d_j^{(B)} := 2\omega \alpha_j^{(B)\dagger} \alpha_j^{(B)} + a \sum_{k=j+1}^{N} (1 + t_j t_k) K_{jk} + b t_j.$$

We call them the Cherednik operators [5, 6, 8, 14]. The Cherednik operators and the exchange and reflection operators satisfy

$$\begin{aligned} &d_l^{(A)} K_l - K_l d_{l+1}^{(A)} = a \qquad d_{l+1}^{(A)} K_l - K_l d_l^{(A)} = -a \\ &[d_l^{(A)}, K_m] = 0 \qquad \text{for} \quad l \neq m, m+1 \end{aligned} \tag{2.5a}$$

$$\begin{aligned} &d_l^{(B)} K_l - K_l d_{l+1}^{(B)} = a(1 + t_l t_{l+1}) \\ &d_{l+1}^{(B)} K_l - K_l d_l^{(B)} = -a(1 + t_l t_{l+1}) \\ &[d_l^{(B)}, K_m] = 0 \qquad \text{for} \quad l \neq m, m+1 \\ &[d_l^{(B)}, t_m] = 0 \end{aligned} \tag{2.5b}$$

where the exchange operators $K_{l,l+1}$ for $l \in \{1, 2, \ldots, N-1\}$ are denoted by K_l. In terms of the Cherednik operators, the Calogero Hamiltonians (2.2) can be expressed as

$$\mathcal{H}^{(A)} = \omega \sum_{l=1}^{N} \left(d_l^{(A)} - \tfrac{1}{2} a(N-1) \right) \tag{2.6a}$$

$$\mathcal{H}^{(B)} = \omega \sum_{l=1}^{N} \left(d_l^{(B)} - a(N-1) - b \right). \tag{2.6b}$$

Thus we conclude that the Cherednik operators $\{d_l^{(A,B)}|l = 1, 2, \dots, N\}$ give the sets of commutative conserved operators of the Calogero models. The last formula in (2.5*b*) and the B_N-Calogero Hamiltonian (2.6*b*) imply that the parity of each variable is a good quantum number of the B_N-Calogero model with distinguishable particles.

3. Non-symmetric multivariable Hermite and Laguerre polynomials

The Cherednik operators define inhomogeneous multivariable polynomials as their joint polynomial eigenfunctions, which are nothing but the non-symmetric multivariable Hermite and Laguerre polynomials that form orthogonal bases of the polynomial ring $\mathbb{C}[x]$ [2,10,28,48].

In order to investigate such polynomial eigenfunctions, we need mathematical preparations for a root system and the associated Weyl group [12]. Let $\check{I} = \{1, 2, \dots, N-1\}$ and $I = \{1, 2, \dots, N\}$ be sets of indices and let V be an N-dimensional real vector space with positive definite bilinear form $\langle\cdot,\cdot\rangle$. We take an orthogonal basis $\{\varepsilon_j | j \in I\}$ of V such that $\langle\varepsilon_j, \varepsilon_k\rangle = \delta_{jk}$. We realize the A_{N-1}-type root system R associated with the simple Lie algebra of type A_{N-1} as

$$R = \{\varepsilon_j - \varepsilon_k | j, k \in I, j \neq k\} (\subset V).$$

A root basis of R is defined by

$$\Pi := \{\alpha_j = \varepsilon_j - \varepsilon_{j+1} | j \in \check{I}\}$$

whose elements are called simple roots. We denote by R_+ the set of positive roots relative to Π and $R_- = -R_+$. The root lattice Q is defined by $Q := \bigoplus_{j\in\check{I}} \mathbb{Z}\alpha_j$ and the positive root lattice Q_+ is defined by replacing $\mathbb{Z}$ with $\mathbb{Z}_{\geqslant 0}$.

We consider a reflection on V with respect to the hyperplane that is orthogonal to a root $\alpha \in R$, and indicate it by $s_\alpha(\mu) := \mu - \langle\alpha^\vee, \mu\rangle\alpha$, where $\alpha^\vee := 2\alpha/\langle\alpha, \alpha\rangle$ is a coroot corresponding to $\alpha \in R$. The reflections $\{s_j := s_{\alpha_j} | \alpha_j \in \Pi\}$ generate the A_{N-1}-type Weyl group W which is isomorphic to the symmetric group $\mathfrak{S}_N$, $W \simeq \mathfrak{S}_N$. For each $w \in W$, we define the following set of positive roots: $R_w := R_+ \cap w^{-1}R_-$. We denote by $\ell(w)$ the length of $w \in W$ defined by $\ell(w) := |R_w|$. When $w \in W$ is written as a product of simple reflections, e.g. $w = s_{j_k} \cdots s_{j_2} s_{j_1}$, the length $\ell(w)$ gives the smallest k for such expressions. We call an expression $w = s_{j_l} \cdots s_{j_2} s_{j_1}$, $l = \ell(w)$, reduced. If we take the above-reduced expression, the set R_w is expressed by

$$R_w = \{\alpha_{j_1}, s_{j_1}(\alpha_{j_2}), \dots, s_{j_1} s_{j_2} \cdots s_{j_{l-1}}(\alpha_{j_l})\}.$$

Though reduced expressions may not be unique for each $w \in W$, it is known that the above set R_w is unique as a set for each $w \in W$ [12].

We introduce lattices $P := \bigoplus_{j\in I} \mathbb{Z}_{\geqslant 0}\varepsilon_j$ and $P_+ := \{\mu = \sum_{j\in I} \mu_j\varepsilon_j \in P | \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N \geqslant 0\}$ whose elements are called a composition and a partition, respectively. The lattice P is W-stable. The degree of the composition and partition is denoted by $|\mu| := \sum_{j\in I} \mu_j$. Let $W(\mu) := \{w(\mu) | w \in W\}$ be the W-orbit of $\mu \in P$. In a W-orbit $W(\mu)$, there exists a unique partition $\mu^+ \in P_+$ such that $\mu = w(\mu^+) \in P$ $(w \in W)$. We define

$$\rho := \tfrac{1}{2}\sum_{\alpha\in R_+} \alpha = \tfrac{1}{2}\sum_{j\in I}(N - 2j + 1)\varepsilon_j \qquad 1^N := \sum_{j\in I} \varepsilon_j$$

$$\delta := \sum_{j\in I}(N - j)\varepsilon_j = \rho + \tfrac{1}{2}(N - 1)1^N.$$

In order to deal with the eigenvalues of the Cherednik operators in terms of the lattice P, we introduce the following operators:

$$d^{(A,B)\lambda} := \sum_{j\in I} \lambda_j d_j^{(A,B)} \qquad t^\lambda := t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_N^{\lambda_N} \qquad \lambda \in P$$

which relate the Cherednik and reflection operators with the lattice P.

We identify the elements of the lattice P with those of the polynomial ring with N variables over $\mathbb{C}$, $x^{\mu} := x_1^{\mu_1} x_2^{\mu_2} \cdots x_N^{\mu_N} \in \mathbb{C}[x]$. Then the action of the coordinate exchange operators $\{K_j | j \in \check{I}\}$ on $\mathbb{C}[x]$ are written as

$$K_j(x^{\mu}) = x^{s_j(\mu)} \qquad \text{for} \quad x^{\mu} \in \mathbb{C}[x].$$

We denote the $(W$-)symmetric and $(W$-)anti-symmetric polynomial rings over $\mathbb{C}$ by $\mathbb{C}[x]^{\pm W}$. On the other hand, the action of the reflection operators on $\mathbb{C}[x]$ is expressed as

$$t_j(x^{\mu}) = (-1)^{\langle \varepsilon_j, \mu \rangle} x^{\mu} \qquad t^{\alpha_j^{\vee}}(x^{\mu}) = (-1)^{\langle \alpha_j^{\vee}, \mu \rangle} x^{\mu} \qquad \text{for} \quad x^{\mu} \in \mathbb{C}[x].$$

We shall use such notations quite often in the following.

We denote the shortest element of W such that $w_{\mu}^{-1}(\mu) \in P_+$ by w_{μ} and define $\rho(\mu) := w_{\mu}(\rho)$ and $\delta(\mu) := w_{\mu}(\delta)$. The definitions of the (monic) non-symmetric multivariable Hermite and Laguerre polynomials, $h_{\mu}^{(A,B)} \in \mathbb{C}[x]$, $\mu \in P$, as the joint eigenvectors for the commutative Cherednik operators $\{d^{(A,B)\lambda}\}$ are given by

$$\begin{aligned} & h_{\mu}^{(A)}(x) = x^{\mu} + \sum_{\substack{\nu \preceq \mu \\ \text{or } |\nu| < |\mu|}} v_{\mu\nu}^{(A)}\left(a, \frac{1}{2\omega}\right) x^{\nu} \\ & d^{(A)\lambda} h_{\mu}^{(A)} = \langle \lambda, \mu + a\rho(\mu) + \tfrac{1}{2}a(N-1)1^N \rangle h_{\mu}^{(A)} = \langle \lambda, \mu + a\delta(\mu) \rangle h_{\mu}^{(A)} \end{aligned} \tag{3.1a}$$

$$\begin{aligned} & h_{\mu}^{(B)}(x) = x^{\mu} + \sum_{\substack{\nu \preceq \mu \\ \text{or } |\nu| < |\mu|}} v_{\mu\nu}^{(B)}\left(a, b, \frac{1}{2\omega}\right) x^{\nu} \\ & d^{(B)\lambda} h_{\mu}^{(B)} = \langle \lambda, \mu + 2a\rho(\mu) + (a(N-1)+b)1^N \rangle h_{\mu}^{(B)} = \langle \lambda, \mu + \rho_k^{(B)}(\mu) \rangle h_{\mu}^{(B)} \end{aligned} \tag{3.1b}$$

where

$$\rho_k^{(B)} := \sum_{j \in I} \big(2a(N-j) + b\big)\varepsilon_j \qquad \rho_k^{(B)}(\mu) := w_{\mu}(\rho_k^{(B)}).$$

The triangularity is defined by the order $\preceq$ on P:

$$\nu \preceq \mu \qquad (\nu, \mu \in P) \Leftrightarrow \begin{cases} \nu^+ \overset{\mathrm{d}}{<} \mu^+ & \nu \notin W(\mu^+) \\ \mu - \nu \in Q_+ & \nu \in W(\mu^+). \end{cases} \tag{3.2}$$

Here, the symbol $\overset{\mathrm{d}}{<}$ denotes the dominance order among partitions

$$\nu \overset{\mathrm{d}}{<} \mu \ (\mu, \nu \in P_+) \Leftrightarrow \mu \neq \lambda \qquad |\mu| = |\nu| \qquad \text{and} \qquad \sum_{k=1}^{l} \nu_k \leqslant \sum_{k=1}^{l} \mu_k$$

for all $l \in I$. We should note that the non-symmetric multivariable Laguerre polynomial is the joint eigenvector of the reflection operators t_j, $j \in I$,

$$t^{\lambda} h_{\mu}^{(B)} = (-1)^{\langle \lambda, \mu \rangle} h_{\mu}^{(B)}$$

and the parity with respect to each variable is a quantum number of the B_N-Calogero models with distinguishable particles. The above formula tells us that the parity of $h_{\mu}^{(B)}$ with respect to a variable x_j is $(-1)^{\mu_j}$.

Since $d^{(A,B)\lambda}$ are Hermitian operators with respect to the inner products (2.4),

$$\langle f, d^{(A,B)\lambda} g \rangle_{(A,B)} = \langle d^{(A,B)\lambda} f, g \rangle_{(A,B)}$$

and all the simultaneous eigenspaces of the Cherednik operators $\{d^{(A,B)\lambda}\}$ are one-dimensional in the sense that the eigenvalues of $\{d^{(A,B)\lambda}\}$ are non-degenerate, it proves that the polynomials $h_{\mu}^{(A,B)}$ are orthogonal with respect to the inner product, i.e. $\langle h_{\mu}^{(A,B)}, h_{\nu}^{(A,B)} \rangle_{(A,B)} =$

$\delta_{\mu,\nu}\|h_\mu^{(A,B)}\|^2$. In fact, the non-symmetric multivariable Hermite and Laguerre polynomials form orthogonal bases in $\mathbb{C}[x]$. We readily confirm that the polynomials (3.1) are generally non-symmetric under exchange of variables $\{x_j\}$.

We should note a connection of the action of the Weyl group and the above definition of the order $\preceq$ (3.2) for compositions in the same W-orbit. Let us compare $s_j(\mu)$ and μ by the order $\preceq$. From the definition of the reflection, we have

$$s_j(\mu) - \mu = -\langle\alpha_j^\vee, \mu\rangle\alpha_j.$$

Thus we conclude $\mu \succeq s_j(\mu)$ if $\langle\alpha_j^\vee, \mu\rangle \geqslant 0$. When one of the reduced expressions of w_μ is given by $s_{j_l}\cdots s_{j_2}s_{j_1}$, $(l = \ell(w_\mu))$, we can confirm the following relation:

$$\begin{aligned} &\mu = \mu^{(l)} \prec \mu^{(l-1)} \prec \cdots \prec \mu^{(0)} = \mu^+ \\ &\mu^{(n)} := s_{j_n}\cdots s_{j_2}s_{j_1}(\mu^+) \qquad n \in \{0, 1, 2, \ldots, l\} \end{aligned} \tag{3.3}$$

using the fact $\langle\alpha^\vee, \mu^+\rangle \geqslant 0$, $^\forall\alpha \in R_{w_\mu} \subseteq R_+$, $\mu^+ \in P_+$. We shall use this relation in the algebraic construction of the polynomials in the next section.

4. Rodrigues formula

We shall present the Rodrigues formulae for the non-symmetric multivariable Hermite and Laguerre polynomials $h_\mu^{(A,B)}$. In order to calculate the square norms of the polynomials, they should be monic in the sense that the coefficients of the top terms x_μ are unity. However, polynomials generated by the Rodrigues formulae presented in our previous works [28, 43] were not monic. Here we show the Rodrigues formulae that generate the monic polynomials.

We introduce the Knop–Sahi operators $\{e^{(A,B)}, e^{(A,B)\dagger}\}$ [17] and the braid operators $\{S_j^{(A,B)} | j \in \check{I}\}$ defined by

$$\begin{aligned} e^{(A,B)} &:= \alpha_1^{(A,B)} K_1 K_2 \cdots K_{N-1} \\ e^{(A,B)\dagger} &= K_{N-1}\cdots K_2 K_1 \alpha_1^{(A,B)\dagger} \end{aligned} \tag{4.1}$$

$$S_j^{(A,B)} := [K_j, d_j^{(A,B)}]. \tag{4.2}$$

The operators $\{e^{(A)}, e^{(A)\dagger}\}$ were first introduced by Baker and Forrester [2]. The Knop–Sahi operators and the braid operators satisfy the following relations:

$$\begin{aligned} &S_j^{(A,B)}S_{j+1}^{(A,B)}S_j^{(A,B)} = S_{j+1}^{(A,B)}S_j^{(A,B)}S_{j+1}^{(A,B)} \qquad \text{for} \quad 1 \leqslant j \leqslant N-2 \\ &S_j^{(A,B)}S_k^{(A,B)} = S_k^{(A,B)}S_j^{(A,B)} \qquad \text{for} \quad |j-k| \geqslant 2 \\ &t_j S_j^{(B)} t_{j+1} S_j^{(B)} = S_j^{(B)} t_{j+1} S_j^{(B)} t_j \\ &S_j^{(A,B)} e^{(A,B)\dagger} = e^{(A,B)\dagger} S_{j+1}^{(A,B)} \qquad \text{for} \quad 1 \leqslant j \leqslant N-2 \\ &S_{N-1}^{(A,B)} (e^{(A,B)\dagger})^2 = (e^{(A,B)\dagger})^2 S_1^{(A,B)} \\ &(S_j^{(A)})^2 = a^2 - (d_j^{(A)} - d_{j+1}^{(A)})^2 \\ &(S_j^{(B)})^2 = 2a^2(1 + t_j t_{j+1}) - (d_j^{(B)} - d_{j+1}^{(B)})^2 \\ &S_j^{(A,B)\dagger} = -S_j^{(A,B)} \qquad e^{(A,B)\dagger} e^{(A,B)} = \frac{1}{2\omega} d_N^{(A,B)} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &S_j^{(A,B)} d^{(A,B)\lambda} = d^{(A,B)s_j(\lambda)} S_j^{(A,B)} \qquad S_j^{(B)} t^\lambda = t^{s_j(\lambda)} S_j^{(B)} \\ &d^{(A,B)\lambda} e^{(A,B)\dagger} = e^{(A,B)\dagger}\big(d^{(A,B)s_1 s_2\cdots s_{N-1}(\lambda)} + \langle\lambda, \varepsilon_N\rangle\big). \end{aligned} \tag{4.4}$$

The first relation in (4.3) is called the braid relation. The relations (4.4) indicates that the operators $\{S_j^{(A,B)\dagger}, e^{(A,B)\dagger}\}$ intertwine the simultaneous eigenspaces of $\{d^{(A,B)\lambda}\}$. We define the raising operators $\{A_\mu^{(A,B)\dagger} | \mu \in P_+\}$ by

$$\begin{aligned} A_\mu^{(A,B)\dagger} &:= (A_1^{(A,B)\dagger})^{\mu_1-\mu_2}(A_2^{(A,B)\dagger})^{\mu_2-\mu_3}\cdots(A_N^{(A,B)\dagger})^{\mu_N} \\ A_j^{(A,B)\dagger} &:= (S_j^{(A,B)}S_{j+1}^{(A,B)}\cdots S_{N-1}^{(A,B)}e^{(A,B)\dagger})^j \qquad \text{for} \quad j \in I. \end{aligned} \tag{4.5}$$

Equations (4.3) and (4.4) yield the following raising operator relations:

$$\begin{aligned} d^{(A,B)\lambda}A_\mu^{(A,B)\dagger} &= A_\mu^{(A,B)\dagger}(d^{(A,B)\lambda} + \langle\lambda, \mu\rangle) \\ t^\lambda A_\mu^{(B)\dagger} &= (-1)^{\langle\lambda,\mu\rangle}A_\mu^{(B)\dagger}t^\lambda \\ \left[A_\mu^{(A,B)\dagger}, A_\nu^{(A,B)\dagger}\right] &= 0 \qquad \text{for} \quad \mu, \nu \in P_+ \end{aligned} \tag{4.6}$$

which lead to the Rodrigues formulae for the non-symmetric multivariable Hermite and Laguerre polynomials that are identified with partitions, $h_\mu^{(A,B)}$, $\mu \in P_+$.

Proposition 4.1 (cf [28,48]). *The non-symmetric multivariable Hermite and Laguerre polynomials $h_\mu^{(A,B)}$ with a partition $\mu \in P_+$ are algebraically constructed by applying the raising operators $A_\mu^{(A,B)\dagger}$ to $h_0^{(A,B)} = 1$,*

$$h_\mu^{(A,B)} = c_\mu^{(A,B)-1}A_\mu^{(A,B)\dagger}h_0^{(A,B)} \tag{4.7}$$

where the coefficients of the top terms $c_\mu^{(A,B)}$ are given by

$$c_\mu^{(A)} := \prod_{\alpha\in R_+}\prod_{l=1}^{\langle\alpha^\vee,\mu\rangle}\left(l + a\langle\alpha^\vee, \rho\rangle\right) \tag{4.8a}$$

$$c_\mu^{(B)} := \prod_{\alpha\in R_+}\prod_{l=1}^{\langle\alpha^\vee,\mu\rangle}\left(l + 2a\langle\alpha^\vee, \rho\rangle\right). \tag{4.8b}$$

To calculate the coefficients, we need to know the action of the coordinate exchange operators on the polynomials.

Lemma 4.2 (cf [3,17]). *Applying the coordinate exchange operators $\{K_j | j \in \check{I}\}$ to the non-symmetric Hermite and Laguerre polynomials $h_\mu^{(A,B)} \in \mathbb{C}[x]$, $(\mu \in P)$, we find that*

$$K_jh_\mu^{(A)} = \begin{cases} \dfrac{a}{\langle\alpha_j^\vee, \mu + a\rho(\mu)\rangle}h_\mu^{(A)} + h_{s_j(\mu)}^{(A)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle < 0 \\ h_\mu^{(A)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle = 0 \\ \dfrac{a}{\langle\alpha_j^\vee, \mu + a\rho(\mu)\rangle}h_\mu^{(A)} + \left(1 - \dfrac{a^2}{\langle\alpha_j^\vee, \mu + a\rho(\mu)\rangle^2}\right)h_{s_j(\mu)}^{(A)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle > 0 \end{cases}$$

$$K_jh_\mu^{(B)} = \begin{cases} \dfrac{a(1 + (-1)^{\langle\alpha_j^\vee,\mu\rangle})}{\langle\alpha_j^\vee, \mu + 2a\rho(\mu)\rangle}h_\mu^{(B)} + h_{s_j(\mu)}^{(B)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle < 0 \\ h_\mu^{(B)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle = 0 \\ \dfrac{a(1 + (-1)^{\langle\alpha_j^\vee,\mu\rangle})}{\langle\alpha_j^\vee, \mu + 2a\rho(\mu)\rangle}h_\mu^{(B)} & \\ + \left(1 - \dfrac{a^2(1 + (-1)^{\langle\alpha_j^\vee,\mu\rangle})^2}{\langle\alpha_j^\vee, \mu + 2a\rho(\mu)\rangle^2}\right)h_{s_j(\mu)}^{(B)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle > 0. \end{cases}$$

Proof. The proof of the lemma is straightforward if one uses the definitions and orthogonality of the polynomials and the commutation relation (2.5). □

Proof of proposition 4.1. Let $\tilde{h}_\mu^{(A,B)} := A_\mu^{(A,B)\dagger} h_0^{(A,B)}$. By a straightforward calculation using (4.6), we can confirm

$$d^{(A)\lambda}\tilde{h}_\mu^{(A)} = A_\mu^{(A)\dagger}(d^{(A)\lambda} + \langle\lambda, \mu\rangle)h_0^{(A)} = \langle\lambda, \mu + a\rho(\mu) + \tfrac{1}{2}a(N-1)1^N\rangle\tilde{h}_\mu^{(A)}$$
$$d^{(B)\lambda}\tilde{h}_\mu^{(B)} = A_\mu^{(B)\dagger}(d^{(B)\lambda} + \langle\lambda, \mu\rangle)h_0^{(B)} = \langle\lambda, \mu + 2a\rho(\mu) + (a(N-1) + b)1^N\rangle\tilde{h}_\mu^{(B)}$$

which are nothing but the second and the forth relations in the definitions of the non-symmetric multivariable Hermite and Laguerre polynomials (3.1). Since all the simultaneous eigenspaces of the Cherednik operators $\{d^{(A,B)\lambda}\}$ are one-dimensional, we can identify $\tilde{h}_\mu^{(A,B)}$ with the non-symmetric multivariable Hermite and Laguerre polynomials $h_\mu^{(A,B)}$, $(\mu \in P_+)$ up to constant multiplicative coefficients $c_\mu^{(A,B)}$.

From the lemma above and the definition of the braid operators (4.2), it is straightforward to compute the action of the braid operators $S_j^{(A,B)}$ on $h_\mu^{(A,B)}$:

$$S_j^{(A)}h_\mu^{(A)} = \begin{cases} \dfrac{a}{\langle\alpha_j^\vee, \mu + a\rho(\mu)\rangle}h_{s_j(\mu)}^{(A)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle < 0 \\ 0 & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle = 0 \\ \dfrac{\langle\alpha_j^\vee, \mu + a\rho(\mu)\rangle^2 - a^2}{\langle\alpha_j^\vee, \mu + a\rho(\mu)\rangle}h_{s_j(\mu)}^{(A)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle > 0 \end{cases} \tag{4.9a}$$

$$S_j^{(B)}h_\mu^{(B)} = \begin{cases} \langle\alpha_j^\vee, \mu + 2a\rho(\mu)\rangle h_{s_j(\mu)}^{(B)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle < 0 \\ 0 & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle = 0 \\ \dfrac{\langle\alpha_j^\vee, \mu + 2a\rho(\mu)\rangle^2 - a^2(1 + (-1)^{\langle\alpha_j^\vee, \mu\rangle})^2}{\langle\alpha_j^\vee, \mu + 2a\rho(\mu)\rangle}h_{s_j(\mu)}^{(B)} & \text{if} \quad \langle\alpha_j^\vee, \mu\rangle > 0. \end{cases} \tag{4.9b}$$

Then we can confirm that the coefficients $c_\mu^{(A,B)}$ are given by (4.8) by calculations using (3.1) and (4.9). □

Since we have the braid operators $S_j^{(A,B)}$ that operate on the polynomials $h_\mu^{(A,B)}$ and generate polynomials $h_{s_j(\mu)}^{(A,B)}$, all we have to do to construct the non-symmetric multivariable Hermite and Laguerre polynomials with a general composition $\mu \in P$ lying in $W(\mu^+)$ is to apply the braid operators to the polynomials $h_{\mu^+}^{(A,B)}$, $\mu^+ \in P_+$.

Proposition 4.3. *Let $w_\mu = s_{j_l}\cdots s_{j_2}s_{j_1}$ be one of the reduced expressions of w_μ and let S_{w_μ} be defined by $S_{w_\mu} := S_{j_l}\cdots S_{j_2}S_{j_1}$. Then the non-symmetric multivariable Hermite and Laguerre polynomials with a composition $\mu \in P$ in the W-orbit of the partition $\mu^+ \in P_+$ are obtained from $h_{\mu^+}^{(A,B)}$ by*

$$h_\mu^{(A,B)} = (c_{w_\mu}^{(A,B)})^{-1}S_{w_\mu}^{(A,B)}h_{\mu^+}^{(A,B)} \tag{4.10}$$

where the coefficients of the top terms $c_{w_\mu}^{(A,B)}$ are expressed as

$$c_{w_\mu}^{(A)} := \prod_{\alpha\in R_{w_\mu}} \frac{\langle\alpha^\vee, \mu^+ + a\rho\rangle^2 - a^2}{\langle\alpha^\vee, \mu^+ + a\rho\rangle} \tag{4.11a}$$

$$c_{w_\mu}^{(B)} := \prod_{\alpha\in R_{w_\mu}} \frac{\langle\alpha^\vee, \mu^+ + 2a\rho\rangle^2 - a^2\big(1 + (-1)^{\langle\alpha^\vee, \mu\rangle}\big)^2}{\langle\alpha^\vee, \mu^+ + 2a\rho\rangle}. \tag{4.11b}$$

Proof. The proposition is verified by (4.4) and (4.9) for $\langle \alpha_j^\vee, \mu \rangle > 0$. □

Combining propositions 4.1 and 4.3, we immediately obtain the Rodrigues formula for the non-symmetric multivariable Hermite and Laguerre polynomials with a general composition $h_\mu^{(A,B)}$, $\mu \in P$.

Theorem 4.4 (Rodrigues formula). *The monic non-symmetric multivariable Hermite and Laguerre polynomials $h_\mu^{(A,B)}$ with a general composition $\mu \in P$ are algebraically obtained by applying the raising operators $A_{\mu^+}^{(A,B)\dagger}$ and the product of braid operators S_{w_μ} to $h_0^{(A,B)} = 1$,*

$$h_\mu^{(A,B)} = (c_{w_\mu}^{(A,B)} c_{\mu^+}^{(A,B)})^{-1} S_{w_\mu}^{(A,B)} A_{\mu^+}^{(A,B)\dagger} h_0^{(A,B)}. \tag{4.12}$$

We note that the corresponding formulae for the eigenstates, $\varphi_\mu^{(A,B)}(x) := h_\mu^{(A,B)}(x)\phi_{\mathrm{g}}^{(A,B)}(x)$, $\mu \in P$, of the original Hamiltonian $\hat{\mathcal{H}}^{(A,B)}$ (2.1) are

$$\begin{aligned}\varphi_\mu^{(A,B)} &= \phi_{\mathrm{g}}^{(A,B)} (c_{w_\mu}^{(A,B)} c_{\mu^+}^{(A,B)})^{-1} S_{w_\mu}^{(A,B)} A_{\mu^+}^{(A,B)\dagger} h_0^{(A,B)} \\ &= (c_{w_\mu}^{(A,B)} c_{\mu^+}^{(A,B)})^{-1} \hat{S}_{w_\mu}^{(A,B)} \hat{A}_{\mu^+}^{(A,B)\dagger} \phi_{\mathrm{g}}^{(A,B)}\end{aligned}$$

where $\hat{S}_{w_\mu}^{(A,B)} := \phi_{\mathrm{g}}^{(A,B)} \circ S_{w_\mu}^{(A,B)} \circ (\phi_{\mathrm{g}}^{(A,B)})^{-1}$ and $\hat{A}_{\mu^+}^{(A,B)\dagger} := \phi_{\mathrm{g}}^{(A,B)} \circ A_{\mu^+}^{(A,B)\dagger} \circ (\phi_{\mathrm{g}}^{(A,B)})^{-1}$.

Now we shall calculate norms of the non-symmetric multivariable Hermite and Laguerre polynomials in an algebraic fashion using the Rodrigues formula. We also use the norms for $h_0^{(A,B)} = 1$:

$$\begin{aligned}\langle h_0^{(A)}, h_0^{(A)} \rangle_{(A)} &= \frac{(2\pi)^{\frac{N}{2}}}{(2\omega)^{\frac{1}{2}N(Na+(1-a))}} \prod_{j\in I} \frac{\Gamma(1+ja)}{\Gamma(1+a)} \\ \langle h_0^{(B)}, h_0^{(B)} \rangle_{(B)} &= \frac{1}{\omega^{N(N-1)a+N(b+\frac{1}{2})}} \prod_{j\in I} \frac{\Gamma(1+ja)\Gamma((j-1)a+b+\frac{1}{2})}{\Gamma(1+a)}\end{aligned}$$

which are proved by certain limits of the Selberg integral [25, 26].

First we shall calculate the square norms of the polynomials with a general composition divided by those with the corresponding partition.

Lemma 4.5 (cf [3, 17]). *For the non-symmetric multivariable Hermite and Laguerre polynomials with a general composition $h_\mu^{(A,B)}$, $\mu \in W(\mu^+)$, $\mu^+ \in P_+$, we have*

$$\langle h_\mu^{(A)}, h_\mu^{(A)} \rangle_{(A)} = \prod_{\alpha \in R_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + a\rho \rangle^2}{\langle \alpha^\vee, \mu^+ + a\rho \rangle^2 - a^2} \langle h_{\mu^+}^{(A)}, h_{\mu^+}^{(A)} \rangle \tag{4.13a}$$

$$\langle h_\mu^{(B)}, h_\mu^{(B)} \rangle_{(B)} = \prod_{\alpha \in R_{w_\mu}} \frac{\langle \alpha^\vee, \mu^+ + 2a\rho \rangle^2}{\langle \alpha^\vee, \mu^+ + 2a\rho \rangle^2 - a^2(1+(-1)^{\langle \alpha^\vee, \mu^+ \rangle})^2} \langle h_{\mu^+}^{(B)}, h_{\mu^+}^{(B)} \rangle \tag{4.13b}$$

which are independent of the choice of reduced expressions of w_μ.

Proof. Due to proposition 4.3, the square norms of the non-symmetric multivariable Hermite and Laguerre polynomials with a general composition $h_\mu^{(A,B)}$, $\mu \in P$ are expressed as

$$\begin{aligned}\langle h_\mu^{(A,B)}, h_\mu^{(A,B)} \rangle_{(A,B)} &= (c_{w_\mu}^{(A,B)})^{-2} \langle S_{w_\mu}^{(A,B)} h_{\mu^+}^{(A,B)}, S_{w_\mu}^{(A,B)} h_{\mu^+}^{(A,B)} \rangle_{(A,B)} \\ &= (c_{w_\mu}^{(A,B)})^{-2} \langle h_{\mu^+}^{(A,B)}, S_{w_\mu}^{(A,B)\dagger} S_{w_\mu}^{(A,B)} h_{\mu^+}^{(A,B)} \rangle_{(A,B)}\end{aligned}$$

where $S_{w_\mu}^{(A,B)\dagger} = (-S_{j_1})(-S_{j_2})\cdots(-S_{j_l})$. Using (4.3), we have

$$\langle h_\mu^{(A)}, h_\mu^{(A)}\rangle_{(A)} = (c_{w_\mu}^{(A)})^{-2}\prod_{n=1}^{l}\left(\langle\alpha_{j_n}^\vee, \mu^{(n-1)} + a\rho(\mu^{(n-1)})\rangle^2 - a^2\right)\langle h_{\mu^+}^{(A)}, h_{\mu^+}^{(A)}\rangle_{(A)}$$

$$\langle h_\mu^{(B)}, h_\mu^{(B)}\rangle_{(B)} = (c_{w_\mu}^{(B)})^{-2}\prod_{n=1}^{l}\left(\langle\alpha_{j_n}^\vee, \mu^{(n-1)} + 2a\rho(\mu^{(n-1)})\rangle^2 - a^2(1+(-1)^{\langle\alpha_{j_n}^\vee,\mu^{(n-1)}\rangle})^2\right)$$
$$\times\langle h_{\mu^+}^{(B)}, h_{\mu^+}^{(B)}\rangle_{(B)}$$

where the sequence of compositions $\{\mu^{(n)}|n = 1, 2, \ldots, l\}$ is defined by a reduced expression of w_μ as in (3.3).

For calculation to proceed, we need a property related to the reflection. For any $s_j \in W$ and $\mu \in P$ such that $\mu \neq s_j(\mu)$, the following formula holds:

$$s_j(\rho(\mu)) = \rho(s_j(\mu)) \tag{4.14}$$

because $w_{s_j(\mu)} = s_j w_\mu$, if $\mu \neq s_j(\mu)$. From the relations above, we can easily verify

$$\langle\alpha_{j_n}^\vee, \mu^{(n-1)}\rangle = \langle s_{j_n}(\alpha_{j_n}^\vee), \mu^{(n-2)}\rangle = \cdots = \langle s_{j_1}s_{j_2}\cdots s_{j_{n-1}}(\alpha_{j_n}^\vee), \mu^+\rangle$$
$$\langle\alpha_{j_n}^\vee, \rho(\mu^{(n-1)})\rangle = \langle s_{j_n}(\alpha_{j_n}^\vee), \rho(\mu^{(n-2)})\rangle = \cdots = \langle s_{j_1}s_{j_2}\cdots s_{j_{n-1}}(\alpha_{j_n}^\vee), \rho\rangle.$$

Here we have used $\rho(\mu^+) = \rho$, $\mu^+ \in P_+$, which follows from the definition of $\rho(\mu)$, $\mu \in P$. Since the set of roots $\{s_{j_1}s_{j_2}\cdots s_{j_{n-1}}(\alpha_{j_n})|n = 1, 2, \ldots, l\}$ is nothing but R_{w_μ} that is uniquely determined by choosing $\mu \in P$, the square norms $\langle h_\mu^{(A,B)}, h_\mu^{(A,B)}\rangle_{(A,B)}$ can be expressed as

$$\langle h_\mu^{(A)}, h_\mu^{(A)}\rangle_{(A)} = (c_{w_\mu}^{(A)})^{-2}\prod_{\alpha\in R_{w_\mu}}\left(\langle\alpha^\vee, \mu^+ + a\rho\rangle^2 - a^2\right)\langle h_{\mu^+}^{(A)}, h_{\mu^+}^{(A)}\rangle_{(A)}$$
$$= \prod_{\alpha\in R_{w_\mu}}\frac{\langle\alpha^\vee, \mu^+ + a\rho\rangle^2}{\langle\alpha^\vee, \mu^+ + a\rho\rangle^2 - a^2}\langle h_{\mu^+}^{(A)}, h_{\mu^+}^{(A)}\rangle_{(A)}$$
$$\langle h_\mu^{(B)}, h_\mu^{(B)}\rangle_{(B)} = (c_{w_\mu}^{(B)})^{-2}\prod_{\alpha\in R_{w_\mu}}\left(\langle\alpha^\vee, \mu^+ + 2a\rho\rangle^2 - a^2(1+(-1)^{\langle\alpha^\vee,\mu^+\rangle})^2\right)\langle h_{\mu^+}^{(B)}, h_{\mu^+}^{(B)}\rangle_{(B)}$$
$$= \prod_{\alpha\in R_{w_\mu}}\frac{\langle\alpha^\vee, \mu^+ + 2a\rho\rangle^2}{\langle\alpha^\vee, \mu^+ + 2a\rho\rangle^2 - a^2(1+(-1)^{\langle\alpha^\vee,\mu^+\rangle})^2}\langle h_{\mu^+}^{(B)}, h_{\mu^+}^{(B)}\rangle_{(B)}$$

which are nothing but the formulae of lemma 4.5. □

The square norms of the polynomials with a partition are summarized as follows.

Proposition 4.6. *The square norms of the non-symmetric multivariable Hermite and Laguerre polynomials* $h_\mu^{(A,B)}$ *with a partition* $\mu \in P_+$ *are given by*

$$\langle h_\mu^{(A)}, h_\mu^{(A)}\rangle_{(A)} = \frac{(2\pi)^{\frac{N}{2}}}{(2\omega)^{\frac{1}{2}N(Na+(1-a))+|\mu|}}\prod_{i\in I}\Gamma\left(\mu_i + a(N-i) + 1\right)$$
$$\times\prod_{\alpha\in R_+}\frac{\Gamma\left(\langle\alpha^\vee, \mu + a\rho\rangle + 1 + a\right)\Gamma\left(\langle\alpha^\vee, \mu + a\rho\rangle + 1 - a\right)}{\Gamma\left(\langle\alpha^\vee, \mu + a\rho\rangle + 1\right)^2} \tag{4.15a}$$

$$\langle h_\mu^{(B)}, h_\mu^{(B)}\rangle_{(B)} = \frac{1}{\omega^{N(N-1)a+N(b+\frac{1}{2})+|\mu|}}$$
$$\times\prod_{i\in I}\Gamma\left(\left[\tfrac{1}{2}(\mu_i+1)\right] + a(N-i) + b + \tfrac{1}{2}\right)\Gamma\left(\left[\tfrac{1}{2}\mu_i\right] + a(N-i) + 1\right)$$

$$\times \prod_{\alpha\in R_+} \frac{\Gamma\big(\big[\frac{1}{2}\langle\alpha^\vee,\mu\rangle\big]+\langle\alpha^\vee,a\rho\rangle+1+a\big)\Gamma\big(\big[\frac{1}{2}\langle\alpha^\vee,\mu\rangle\big]+\langle\alpha^\vee,a\rho\rangle+1-a\big)}{\Gamma\big(\big[\frac{1}{2}\langle\alpha^\vee,\mu\rangle\big]+\langle\alpha^\vee,a\rho\rangle+1\big)^2} \tag{4.15b}$$

where the Gauss's symbol $[x]$ *means the maximum integer that is less than* x.

Proof. In order to calculate the square norms of the polynomials with partitions $\mu \in P_+$ algebraically, we use the following relations:

$$A_j^{(A)}A_j^{(A)\dagger} = \frac{1}{2\omega}\prod_{k=1}^{j}(d_k^{(A)}+1)\prod_{l=j+1}^{N}\big((d_k^{(A)}-d_l^{(A)}+1)^2-a^2\big)$$

$$A_j^{(B)}A_j^{(B)\dagger} = \frac{1}{2\omega}\prod_{k=1}^{j}(d_k^{(B)}+bt_k+1)\prod_{l=j+1}^{N}\big((d_k^{(B)}-d_l^{(B)}+1)^2-2a^2(1-t_kt_l)\big)$$

which can be verified by (4.3)–(4.5), and the defining relations of the non-symmetric multivariable Hermite and Laguerre polynomials (3.1). Then we have

$$\begin{aligned}
\langle h_\mu^{(A)},h_\mu^{(A)}\rangle_{(A)} &= (2\omega)^{-|\mu|}(c_\mu^{(A)})^{-2}\prod_{i=1}^{N}\prod_{j=1}^{i}\prod_{k=1}^{\mu_i-\mu_{i+1}}\big(\mu_i-k+1+a(N-j)\big)\\
&\quad\times\prod_{m=i+1}^{N}\big((\mu_i-\mu_m-k+1+a(m-j))^2-a^2\big)\langle h_0^{(A)},h_0^{(A)}\rangle_{(A)}\\
&= (2\omega)^{-|\mu|}(c_\mu^{(A)})^{-2}\langle h_0^{(A)},h_0^{(A)}\rangle_{(A)}\prod_{i\in I}\prod_{j=1}^{\mu_i}\big(\mu_i-j+1+a(N-i)\big)\\
&\quad\times\prod_{\alpha\in R_+}\prod_{k=1}^{\langle\alpha^\vee,\mu\rangle}\big(\langle\alpha^\vee,\mu+a\rho\rangle-k+1+a\big)\big(\langle\alpha^\vee,\mu+a\rho\rangle-k+1-a\big)\\
&= \frac{(2\pi)^{\frac{N}{2}}}{(2\omega)^{\frac{1}{2}N(Na+(1-a))+|\mu|}}\prod_{i\in I}\Gamma\big(\mu_i+a(N-i)+1\big)\\
&\quad\times\prod_{\alpha\in R_+}\frac{\Gamma\big(\langle\alpha^\vee,\mu+a\rho\rangle+1+a\big)\Gamma\big(\langle\alpha^\vee,\mu+a\rho\rangle+1-a\big)}{\Gamma\big(\langle\alpha^\vee,\mu+a\rho\rangle+1\big)^2}
\end{aligned}$$

$$\begin{aligned}
\langle h_\mu^{(B)},h_\mu^{(B)}\rangle_{(B)} &= (2\omega)^{-|\mu|}(c_\mu^{(B)})^{-2}\\
&\quad\times\prod_{i=1}^{N}\prod_{j=1}^{i}\prod_{k=1}^{\mu_i-\mu_{i+1}}\big(\mu_i-k+1+2a(N-j)+b(1-(-1)^{\mu_i-k+1})\big)\\
&\quad\times\prod_{m=i+1}^{N}\big((\mu_i-\mu_m-k+1+2a(m-j))^2-2a^2(1+(-1)^{\mu_i-\mu_m-k+1})\big)\\
&\quad\times\langle h_0^{(B)},h_0^{(B)}\rangle_{(B)}\\
&= (2\omega)^{-|\mu|}(c_\mu^{(B)})^{-2}\langle h_0^{(B)},h_0^{(B)}\rangle_{(B)}\\
&\quad\times\prod_{i\in I}\prod_{j=1}^{\mu_i}\big(\mu_i-j+1+2a(N-i)+b(1-(-1)^{\mu_i-j+1})\big)\\
&\quad\times\prod_{\alpha\in R_+}\prod_{k=1}^{\langle\alpha^\vee,\mu\rangle}\big(\langle\alpha^\vee,\mu+2a\rho\rangle-k+1+a(1+(-1)^{\langle\alpha^\vee,\mu\rangle-k+1})\big)
\end{aligned}$$

$$\times\big(\langle\alpha^\vee,\mu+2a\rho\rangle-k+1-a(1+(-1)^{\langle\alpha^\vee,\mu\rangle-k+1})\big)$$

$$=\frac{1}{\omega^{N(N-1)a+N(b+\frac{1}{2})+|\mu|}}\prod_{i\in I}\Gamma\big(\big[\tfrac{1}{2}(\mu_i+1)\big]+a(N-i)+b+\tfrac{1}{2}\big)$$

$$\times\Gamma\big(\big[\tfrac{1}{2}\mu_i\big]+a(N-i)+1\big)$$

$$\times\prod_{\alpha\in R_+}\frac{\Gamma\big(\big[\frac{1}{2}\langle\alpha^\vee,\mu\rangle\big]+\langle\alpha^\vee,a\rho\rangle+1+a\big)\Gamma\big(\big[\frac{1}{2}\langle\alpha^\vee,\mu\rangle\big]+\langle\alpha^\vee,a\rho\rangle+1-a\big)}{\Gamma\big(\big[\frac{1}{2}\langle\alpha^\vee,\mu\rangle\big]+\langle\alpha^\vee,a\rho\rangle+1\big)^2}$$

which prove proposition 4.6. □

Finally, we obtain the formulae for the square norms of the non-symmetric multivariable Hermite and Laguerre polynomials with a general composition including orthogonality.

Theorem 4.7. *For the non-symmetric multivariable Hermite and Laguerre polynomials $h_\mu^{(A,B)}$ with a general composition $\mu\in W(\mu^+)$, $\mu^+\in P_+$, we have*

$$\langle h_\mu^{(A)},h_\nu^{(A)}\rangle_{(A)}=\delta_{\mu,\nu}\frac{(2\pi)^{\frac{N}{2}}}{(2\omega)^{\frac{1}{2}N(Na+(1-a))+|\mu|}}\prod_{\beta\in R_{w_\mu}}\frac{\langle\beta^\vee,\mu^++a\rho\rangle^2}{\langle\beta^\vee,\mu^++a\rho\rangle^2-a^2}$$

$$\times\prod_{i\in I}\Gamma\big(\mu_i^++a(N-i)+1\big)$$

$$\times\prod_{\alpha\in R_+}\frac{\Gamma\big(\langle\alpha^\vee,\mu^++a\rho\rangle+1+a\big)\Gamma\big(\langle\alpha^\vee,\mu^++a\rho\rangle+1-a\big)}{\Gamma\big(\langle\alpha^\vee,\mu^++a\rho\rangle+1\big)^2}\tag{4.16a}$$

$$\langle h_\mu^{(B)},h_\nu^{(B)}\rangle_{(B)}=\delta_{\mu,\nu}\frac{1}{\omega^{N(N-1)a+N(b+\frac{1}{2})+|\mu|}}\prod_{\beta\in R_{w_\mu}}\frac{\langle\beta^\vee,\mu^++2a\rho\rangle^2}{\langle\beta^\vee,\mu^++2a\rho\rangle^2-a^2(1+(-1)^{\langle\beta^\vee,\mu^+\rangle})^2}$$

$$\times\prod_{i\in I}\Gamma\Big(\Big[\tfrac{1}{2}(\mu_i^++1)\Big]+a(N-i)+b+\tfrac{1}{2}\Big)\Gamma\Big(\Big[\tfrac{1}{2}\mu_i^+\Big]+a(N-i)+1\Big)$$

$$\times\prod_{\alpha\in R_+}\frac{\Gamma\Big(\Big[\frac{1}{2}\langle\alpha^\vee,\mu^+\rangle\Big]+\langle\alpha^\vee,a\rho\rangle+1+a\Big)\Gamma\Big(\Big[\frac{1}{2}\langle\alpha^\vee,\mu^+\rangle\Big]+\langle\alpha^\vee,a\rho\rangle+1-a\Big)}{\Gamma\Big(\Big[\frac{1}{2}\langle\alpha^\vee,\mu^+\rangle\Big]+\langle\alpha^\vee,a\rho\rangle+1\Big)^2}.\tag{4.16b}$$

In terms of the eigenstates of the original Calogero Hamiltonian, the above orthogonality relations are expressed by $(\varphi_\mu^{(A,B)},\varphi_\nu^{(A,B)})=\langle h_\mu^{(A,B)},h_\nu^{(A,B)}\rangle_{(A,B)}$.

Thus we have presented an algebraic method that enables us to obtain all the non-symmetric multivariable Hermite and Laguerre polynomials with general compositions and their square norms.

5. Symmetrization and anti-symmetrization

We can readily confirm that the non-symmetric multivariable Hermite and Laguerre polynomials with compositions μ lying in the same W-orbit of the partition μ^+ have the same eigenvalue of the Hamiltonians (2.2),

$$\mathcal{H}^{(A,B)}h_\mu^{(A,B)}=\omega|\mu^+|h_\mu^{(A,B)}\qquad\text{for}\quad\mu\in W(\mu^+)\quad\mu^+\in P_+.$$

More generally, the polynomials with compositions $\mu\in W(\mu^+)$ have the same eigenvalue of an arbitrary symmetric polynomial, e.g. any of the power sums, of the Cherednik operators.

Thus any linear combinations of $h_\mu^{(A,B)}$, $\mu \in W(\mu^+)$, $\mu^+ \in P_+$ are eigenfunctions of the Calogero Hamiltonians $\mathcal{H}^{(A,B)}$ and all of their higher-order conserved operators.

Among all such linear combinations, we shall deal with symmetric and anti-symmetric eigenvectors of the Calogero Hamiltonians in $\mathbb{C}[x]^{\pm W}$ that respectively correspond to the bosonic and the fermionic eigenstates of the models. We symmetrize and anti-symmetrize non-symmetric eigenvectors, but our formulation does not use the symmetrizer or the anti-symmetrizer [2] which makes the coefficients of the top terms differ from unity. To describe the anti-symmetric eigenvectors, we introduce sublattices of P_+ such as $P_+{+}\delta := \{\mu{+}\delta | \mu \in P_+\}$ and so forth. Other sublattices of P_+ in what follows are defined in a similar way. We notice that, for the B_N-case, the parity with respect to each variable is restricted to even or odd since the symmetric and anti-symmetric eigenvectors are eigenvectors of the reflection operators $\{t_j | j \in I\}$ at the same time.

Theorem 5.1. *Let $H_{\mu^+}^{(A,B)+}$, $(\mu^+ \in P_+)$, $H_{\mu^+}^{(A)-}$, $(\mu^+ \in P_+ + \delta)$ and $H_{\mu^+}^{(B)-}$, $(\mu^+ \in P_+ + 2\delta)$ be the following linear combinations of the corresponding non-symmetric polynomials with compositions $\mu \in W(\mu^+)$:*

$$H_{\mu^+}^{(A,B)\pm} = \sum_{\mu \in W(\mu^+)} b_{\mu^+\mu}^{(A,B)\pm} h_\mu^{(A,B)} \tag{5.1}$$

whose coefficients are

$$b_{\mu^+\mu}^{(A)\pm} = \prod_{\alpha \in R_{w_\mu}} \pm \frac{\langle \alpha, \mu^+ + a\rho\rangle \mp a}{\langle \alpha, \mu^+ + a\rho\rangle} \qquad b_{\mu^+\mu}^{(B)\pm} = \prod_{\alpha \in R_{w_\mu}} \pm \frac{\langle \alpha, \mu^+ + 2a\rho\rangle \mp 2a}{\langle \alpha, \mu^+ + 2a\rho\rangle}. \tag{5.2}$$

Then we find $H_{\mu^+}^{(A,B)\pm} \in \mathbb{C}[x]^{\pm W}$, which we call the symmetric and anti-symmetric multivariable Hermite and Laguerre polynomials, respectively.

Proof. We consider $H_{\mu^+}^{(A,B)\pm}$ of the forms (5.1). By requiring $H_{\mu^+}^{(A,B)\pm} \in \mathbb{C}[x]^{\pm W}$ and $b_{\mu^+\mu^+}^{(A,B)\pm} = 1$, which is clearly equivalent to the requirement that the coefficients of the top symmetrized monomial are unity, the coefficients $b_{\mu^+\mu}^{(A,B)\pm}$ are uniquely determined. The proofs of the above two formulae are almost the same and we shall only show a proof for $b_{\mu^+\mu}^{(B)\pm}$. Note that, for the B_N-case, $\mu^+ \in 2P_+$ (all even) or $\mu^+ \in 2P_+ + 1^N$ (all odd) so that the parities of all the variables are the same. From lemma 4.2, we have

$$b_{\mu^+\mu}^{(B)\pm} h_\mu^{(B)} + b_{\mu^+ s_j(\mu)}^{(B)\pm} h_{s_j(\mu)}^{(B)} = \pm K_j \big(b_{\mu^+\mu}^{(B)\pm} h_\mu^{(B)} + b_{\mu^+ s_j(\mu)}^{(B)\pm} h_{s_j(\mu)}^{(B)} \big).$$

Without loss of generality, we may assume $\mu \succ s_j(\mu)$ (i.e. $\langle \alpha_j^\vee, \mu\rangle > 0$) since the case $\mu = s_j(\mu)$ (i.e. $\langle \alpha_j^\vee, \mu\rangle = 0$) is trivial. Then the above relation is rewritten as

$$\begin{aligned} b_{\mu^+\mu}^{(B)\pm} h_\mu^{(B)} + b_{\mu^+ s_j(\mu)}^{(B)\pm} h_{s_j(\mu)}^{(B)} = \pm \Bigg(&\left(\frac{2a}{\langle \alpha_j^\vee, \mu + 2a\rho(\mu)\rangle} b_{\mu^+\mu}^{(B)\pm} + b_{\mu^+ s_j(\mu)}^{(B)\pm} \right) h_\mu^{(B)} \\ &+ \left(\frac{\langle \alpha_j^\vee, \mu + 2a\rho(\mu)\rangle^2 - 4a^2}{\langle \alpha_j^\vee, \mu + 2a\rho(\mu)\rangle^2} b_{\mu^+\mu}^{(B)\pm} - \frac{2a}{\langle \alpha_j^\vee, \mu + 2a\rho(\mu)\rangle} b_{\mu^+ s_j(\mu)}^{(B)\pm} \right) h_{s_j(\mu)}^{(B)} \Bigg) \end{aligned}$$

where we have used $1 + (-1)^{\langle \alpha_j^\vee, \mu\rangle} = 2$ for $\mu \in W(\mu^+)$ with $\mu^+ \in 2P_+$ or $\mu^+ \in 2P_+ + 1^N$. Thus we obtain

$$\frac{b_{\mu^+ s_j(\mu)}^{(B)\pm}}{b_{\mu^+\mu}^{(B)\pm}} = \pm \frac{\langle \alpha_j^\vee, \mu + 2a\rho(\mu)\rangle \mp 2a}{\langle \alpha_j^\vee, \mu + 2a\rho(\mu)\rangle} \qquad \text{for} \quad \mu \prec s_j(\mu). \tag{5.3}$$

Let $s_{j_l}\cdots s_{j_2}s_{j_1}$ and $\{\mu^{(n)}\in W(\mu^+)|n=1,2,\ldots,l\}$ be a reduced expression of w_μ and a sequence of compositions as have been given in (3.3). Iterated use of the recursion relation of $b^{(B)\pm}_{\mu^+\mu}$ (5.3) yields

$$b^{(B)\pm}_{\mu^+\mu}=b^{(B)\pm}_{\mu^+\mu^+}\prod_{n=1}^{l}\frac{b^{(B)\pm}_{\mu^+\mu^{(n)}}}{b^{(B)\pm}_{\mu^+\mu^{(n-1)}}}=\prod_{n=1}^{l}\pm\frac{\langle\alpha^\vee_{j_n},\mu^{(n-1)}+2a\rho(\mu^{(n-1)})\rangle\mp 2a}{\langle\alpha^\vee_{j_n},\mu^{(n-1)}+2a\rho(\mu^{(n-1)})\rangle}$$

where $b^{(B)\pm}_{\mu^+\mu^+}=1$ should be noted. By use of equation (4.15), the above formula is cast into the following form:

$$b^{(B)\pm}_{\mu^+\mu}=\prod_{n=1}^{l}\pm\frac{\langle s_{j_1}s_{j_2}\cdots s_{j_{n-1}}(\alpha^\vee_{j_n}),\mu^++2a\rho\rangle\mp 2a}{\langle s_{j_1}s_{j_2}\cdots s_{j_{n-1}}(\alpha^\vee_{j_n}),\mu^++2a\rho\rangle}. \tag{5.4}$$

Recalling the fact $R_{w_\mu}=\{s_{j_1}s_{j_2}\cdots s_{j_{n-1}}(\alpha_{j_n})|n=1,2,\ldots,l\}$, we verify that (5.4) is nothing but $b^{(B)\pm}_{\mu^+\mu}$ in the theorem above. □

The symmetric multivariable Hermite and Laguerre polynomials are the same as those discussed in [1, 14, 44–47, 49]. There are several equivalent conditions to characterize these symmetric polynomials. For instance, triangularity in $\mathbb{C}[x]^W$ and orthogonality with respect to $\langle\cdot,\cdot\rangle_{(A,B)}$ characterize the symmetric multivariable Hermite and Laguerre polynomials up to a constant factor. However, we have implicitly taken another way of characterization. Those polynomials are identified by polynomial parts of eigenstates for all conserved operators of the Calogero models with bosonic particles. We note

$$\begin{aligned}&K_jH^{(A,B)\pm}_\mu=\pm H^{(A,B)\pm}_\mu\\&t_jH^{(B)\pm}_\mu=\begin{cases}H^{(B)\pm}_\mu & \text{for}\quad \mu\in 2P_+\\ -H^{(B)\pm}_\mu & \text{for}\quad \mu\in 2P_++1^N.\end{cases}\end{aligned} \tag{5.5}$$

Therefore, on such (anti-)symmetric functions (with all even or all odd parity for the B_N-case) multiplied by the reference states $\phi^{(A,B)}_{\mathrm{g}}$, the Calogero Hamiltonians (2.1) with distinguishable particles reduce to

$$\hat{\mathcal{H}}^{(A)\pm}(a)=\frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2}+\omega^2x_j^2\right)+\frac{1}{2}\sum_{\substack{j,k=1\\j\neq k}}^{N}\frac{a^2\mp a}{(x_j-x_k)^2}$$

$$\hat{\mathcal{H}}^{(B)\pm,\pm}(a,b)=\frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2}+\omega^2x_j^2+\frac{b^2\mp b}{x_j^2}\right)+\frac{1}{2}\sum_{\substack{j,k=1\\j\neq k}}^{N}\left(\frac{a^2\mp a}{(x_j-x_k)^2}+\frac{a^2\mp a}{(x_j+x_k)^2}\right)$$

which are the Calogero models with indistinguishable (bosonic or fermionic) particles. The first $\pm$ superscripts on the Hamiltonians correspond to the double sign $\mp$ before the coupling parameter a which denote, respectively, the symmetric and anti-symmetric cases. The second $\pm$ superscripts on the B_N-Hamiltonian correspond to the double sign $\mp$ before the coupling parameter b which denote, respectively, all the even and odd cases. The Calogero models $\hat{\mathcal{H}}^{(A)\pm}(a)$ and $\hat{\mathcal{H}}^{(B)\pm,\pm}(a,b)$ are diagonalized by $\Phi^{(A,B)\pm}_\mu(x):=H^{(A,B)\pm}_\mu(x)\phi^{(A,B)}_{\mathrm{g}}(x)\in\mathbb{C}[x]^{\pm W}\phi^{(A,B)}_{\mathrm{g}}$,

$$\begin{aligned}&\hat{\mathcal{H}}^{(A)\pm}(a)\Phi^{(A)\pm}_\mu=(\omega|\mu|+E^{(A)}_{\mathrm{g}})\Phi^{(A)\pm}_\mu\\&\hat{\mathcal{H}}^{(B)\pm,\pm}(a,b)\Phi^{(B)\pm}_\mu=(\omega|\mu|+E^{(B)}_{\mathrm{g}})\Phi^{(B)\pm}_\mu\end{aligned}$$

where the partitions μ should be chosen from the appropriate sublattices of P_+ corresponding to symmetries and parties.

The above Hamiltonians with indistinguishable particles are related to each other by the following shifts of the parameters $a, b \ldots \in \mathbb{R}_{\geqslant 0}$,

$$\hat{\mathcal{H}}^{(A)+}(a+1) = \hat{\mathcal{H}}^{(A)-}(a)$$
$$\hat{\mathcal{H}}^{(B)+,+}(a+1, b+1) = \hat{\mathcal{H}}^{(B)-,+}(a, b+1) = \hat{\mathcal{H}}^{(B)+,-}(a+1, b) = \hat{\mathcal{H}}^{(B)-,-}(a, b).$$

Thus each Hamiltonian with indistinguishable particles has both bosonic and fermionic eigenstates with the appropriate shifts of the parameters in the eigenstates. Directing our attention to polynomial parts of the eigenstates, we notice that the symmetric and anti-symmetric polynomials (with all even or all odd parties for the Laguerre case) are mutually related by

$$\Delta(x) H^{(A)+}_{\mu}(x; a+1) = H^{(A)-}_{\mu+\delta}(x; a) \qquad \mu \in P_+$$
$$\Delta_1(x)\Delta_2(x) H^{(B)+}_{\mu}(x; a+1, b+1) = \Delta_2(x) H^{(B)-}_{\mu+2\delta}(x; a, b+1)$$
$$= \Delta_1(x) H^{(B)+}_{\mu+1^N}(x; a+1, b) = H^{(B)-}_{\mu+2\delta+1^N}(x; a, b) \qquad \mu \in 2P_+$$

where $\Delta(x) := \prod_{\substack{i,j\in I\\ i<j}}(x_i - x_j)$, $\Delta_1(x) := \prod_{\substack{i,j\in I\\ i<j}}(x_i^2 - x_j^2)$ and $\Delta_2(x) := \prod_{i\in I} x_i$. These results for the symmetric or anti-symmetric multivariable Hermite and Laguerre polynomials correspond to those for the Jack and the Macdonald polynomials [7]. The above results mean that the difference between the bosonic and fermionic eigenstates of the Hamiltonians $\hat{\mathcal{H}}^{(A)\pm}(a)$ and $\hat{\mathcal{H}}^{(B)\pm,\pm}(a, b)$, i.e. the products of the reference states and the polynomial parts, comes from the differing choice of the sign of the difference products, $\Delta(x)$ and $\Delta_1(x)$. Similarly, the difference in the parity of the eigenstates of $\hat{\mathcal{H}}^{(B)\pm,\pm}(a, b)$ comes from that of the choice of the sign of $\Delta_2(x)$. To be brief, the statistics of the indistinguishable particles (the symmetry of the eigenstates) and the parity of the eigenstates of the Calogero Hamiltonians, $\hat{\mathcal{H}}^{(A)\pm}(a)$ and $\hat{\mathcal{H}}^{(B)\pm,\pm}(a, b)$, are respectively determined only by the choice of the sign of the difference products, $\Delta(x)$ and $\Delta_1(x)$, and the product $\Delta_2(x)$. We note that similar shifts of the parameter a among the symmetric polynomials are realized by operation of the shift operators on $H^{(A)+}_{\mu}$ and $H^{(B)+}_{2\mu}$, $\mu \in P_+$. The shift operators give recursion relations of the square norms of the symmetric polynomials with respect to the parameter a [14, 32].

From the square norms of the non-symmetric polynomials $\langle h^{(A,B)}_{\mu}, h^{(A,B)}_{\mu}\rangle_{(A,B)}$ and the coefficients $b^{(A,B)\pm}_{\mu^+\mu}$, we shall evaluate the square norms of the (anti-)symmetric eigenfunctions. To prove the formula of the square norms, we need the following lemma.

Lemma 5.2. *For $\mu \in P_+$, we have an identity,*

$$\sum_{\nu\in W(\mu)} \prod_{\alpha\in R_{w_\nu}} \frac{\langle \alpha^\vee, \mu + a\rho\rangle \mp a}{\langle \alpha^\vee, \mu + a\rho\rangle \pm a} = N! \prod_{\alpha\in R_+} \frac{\langle \alpha^\vee, \mu + a\rho\rangle}{\langle \alpha^\vee, \mu + a\rho\rangle \pm a}.$$

The above lemma is proved by use of an expression of the Poincaré polynomials [24, 30] in the appendix.

Theorem 5.3. *Let $\mu \in P_+$ for $H^{(A)+}_{\mu}$, $\mu \in P_+ + \delta$ for $H^{(A)-}_{\mu}$, $\mu \in 2P_+$ or $2P_+ + 1^N$ for $H^{(B)+}_{\mu}$ and $\mu \in 2(P_+ + \delta)$ or $2(P_+ + \delta) + 1^N$ for $H^{(B)-}_{\mu}$. The square norms of the (anti-)symmetric multivariable Hermite and Laguerre polynomials including orthogonality are presented by*

$$\langle H^{(A)\pm}_{\mu}, H^{(A)\pm}_{\nu}\rangle_{(A)} = \delta_{\mu,\nu} \frac{(2\pi)^{\frac{N}{2}} N!}{(2\omega)^{\frac{1}{2}N(Na+(1-a))+|\mu|}} \prod_{j\in I} \Gamma\big(\mu_j + a(N-j) + 1\big)$$
$$\times \prod_{\alpha\in R_+} \frac{\Gamma\big(\langle \alpha^\vee, \mu + a\rho\rangle + 1 \mp a\big)\Gamma\big(\langle \alpha^\vee, \mu + a\rho\rangle \pm a\big)}{\Gamma\big(\langle \alpha^\vee, \mu + a\rho\rangle + 1\big)\Gamma\big(\langle \alpha^\vee, \mu + a\rho\rangle\big)} \qquad (5.6a)$$

$$\langle H_\mu^{(B)\pm}, H_\nu^{(B)\pm}\rangle_{(B)} = \delta_{\mu,\nu}\frac{N!}{\omega^{N(N-1)a+N(b+\frac{1}{2})+|\mu|}} \times \prod_{j\in I}\Gamma\left(\left[\tfrac{1}{2}(\mu_j+1)\right]+a(N-j)+b+\tfrac{1}{2}\right)\Gamma\left(\left[\tfrac{1}{2}\mu_j\right]+a(N-j)+1\right) \times \prod_{\alpha\in R_+}\frac{\Gamma\left(\langle\alpha^\vee, \frac{1}{2}\mu+a\rho\rangle+1\mp a\right)\Gamma\left(\langle\alpha^\vee, \frac{1}{2}\mu+a\rho\rangle\pm a\right)}{\Gamma\left(\langle\alpha^\vee, \frac{1}{2}\mu+a\rho\rangle+1\right)\Gamma\left(\langle\alpha^\vee, \frac{1}{2}\mu+a\rho\rangle\right)}. \qquad (5.6b)$$

Proof. The orthogonality follows from that for the non-symmetric case. The square norms are straightforwardly calculated from lemma 4.5, proposition 5.1 and lemma 5.2. □

In terms of the (anti-)symmetric eigenstates of the Hamiltonians $\hat{\mathcal{H}}^{(A)\pm}(a)$ and $\hat{\mathcal{H}}^{(B)\pm,\pm}(a,b)$, the above formulae are expressed by $(\Phi_\mu^{(A)\pm}, \Phi_\nu^{(A)\pm}) = \langle H_\mu^{(A)\pm}, H_\nu^{(A)\pm}\rangle_{(A)}$ and $(\Phi_\mu^{(B)\pm}, \Phi_\nu^{(B)\pm}) = \langle H_\mu^{(B)\pm}, H_\nu^{(B)\pm}\rangle_{(B)}$. We remark that the square norms of the cases $H_\mu^{(A)+}$, $\mu \in P_+$ and $H_\mu^{(B)+}$, $\mu \in 2P_+$ were calculated in [1, 14, 49] by use of limiting procedure or shift operators, which are different from our approach.

6. Summary

We have presented the Rodrigues formulae for the monic non-symmetric multivariable Hermite and Laguerre polynomials that give the non-symmetric orthogonal bases of the A_{N-1}- and B_N-Calogero models with distinguishable particles. The square norms of the above non-symmetric polynomials have been algebraically calculated by employing a language of a root system of a finite-dimensional simple Lie algebra. Through symmetrization and anti-symmetrization, we have constructed the bosonic and fermionic eigenstates of the Calogero models. The square norms of the bosonic and fermionic eigenstates are calculated from those of their non-symmetric counterparts with the aid of an identity derived by the Poincaré polynomials.

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Appendix. Proof of lemma 5.2

We present a proof of lemma 5.2 in the symmetric case following our previous paper [30]. The Poincaré polynomial is an invariant polynomial which shows remarkable properties of the Weyl group W [12]. They are defined by

$$\mathcal{W}(t) = \sum_{w\in W}\prod_{\alpha\in R_w} t_\alpha$$

where $\{t_\alpha | \alpha \in R\}$ are W-invariant indeterminates, i.e. $t_\alpha = t_{w(\alpha)}$ for $w \in W$. For the Weyl group of type A_{N-1}, in which all the indeterminates are equal $t_\alpha = t$, we have

$$\mathcal{W}(t) = \sum_{w\in W} t^{\ell(w)}.$$

In what follows, we consider only the A_{N-1}-case. We denote by $\mathbb{K}$ the field of rational functions over $\mathbb{C}$ in square-roots of indeterminates $\{t\}$. To investigate the Poincaré polynomials, Macdonald proved the following identity [24]:

Theorem A.1 (Macdonald).

$$\mathcal{W}(t) = \sum_{w\in W} \prod_{\alpha\in R_+} \frac{1 - t x^{w(\alpha^\vee)}}{1 - x^{w(\alpha^\vee)}}. \tag{A.1}$$

Lemma A.2. *Let* $\mu \in P_+$. *We have*

$$\sum_{\nu\in W(\mu)} \prod_{\alpha\in R_{w_\nu}} \frac{t(1 - t^{-1} q^{\langle\alpha^\vee,\mu+a\rho\rangle})}{1 - tq^{\langle\alpha^\vee,\mu+a\rho\rangle}} = \mathcal{W}(t) \prod_{\alpha\in R_+} \frac{1 - q^{\langle\alpha^\vee,\mu+a\rho\rangle}}{1 - tq^{\langle\alpha^\vee,\mu+a\rho\rangle}}. \tag{A.2}$$

Proof. Define a lattice $Q^\vee$ by $Q^\vee := \bigoplus_{j\in\check{I}} \mathbb{Z}\alpha_j^\vee$. There exists a $\mathbb{K}$-homomorphism $\varphi : \mathbb{K}[Q^\vee] \to \mathbb{K}$ defined by

$$\varphi : x^{\alpha_i^\vee} \mapsto q^{\langle\alpha_i^\vee,\mu+a\rho\rangle} \qquad \text{for} \quad i \in \check{I}.$$

Since $\mathcal{W}(t) \in \mathbb{K}[Q^\vee]$ does not depend on $\{x^{\alpha_i^\vee}\}$ as (A.1), we have

$$\begin{aligned}
\varphi\big(\mathcal{W}(t)\big) &= \mathcal{W}(t) \\
&= \sum_{w\in W} \prod_{\alpha\in R_+} \varphi\left(\frac{1 - t x^{w(\alpha^\vee)}}{1 - x^{w(\alpha^\vee)}}\right) \\
&= \sum_{w\in W} \prod_{\alpha\in R_+} \frac{1 - tq^{\langle w(\alpha^\vee),\mu+a\rho\rangle}}{1 - q^{\langle w(\alpha^\vee),\mu+a\rho\rangle}} \\
&= \frac{\sum_{w\in W} \prod_{\alpha\in R_w} (t - q^{\langle\alpha^\vee,\mu+a\rho\rangle}) \prod_{\alpha\in R_+\setminus R_w} (1 - tq^{\langle\alpha^\vee,\mu+a\rho\rangle})}{\prod_{\alpha\in R_+} (1 - q^{\langle\alpha^\vee,\mu+a\rho\rangle})} \\
&= \prod_{\alpha\in R_+} \frac{1 - tq^{\langle\alpha^\vee,\mu+a\rho\rangle}}{1 - q^{\langle\alpha^\vee,\mu+a\rho\rangle}} \sum_{w\in W} \prod_{\alpha\in R_w} \frac{t(1 - t^{-1}q^{\langle\alpha^\vee,\mu+a\rho\rangle})}{1 - tq^{\langle\alpha^\vee,\mu+a\rho\rangle}}.
\end{aligned}$$

Thus we obtain the following relation:

$$\sum_{w\in W} \prod_{\alpha\in R_w} \frac{t(1 - t^{-1} q^{\langle\alpha^\vee,\mu+a\rho\rangle})}{1 - tq^{\langle\alpha^\vee,\mu+a\rho\rangle}} = \mathcal{W}(t) \prod_{\alpha\in R_+} \frac{1 - q^{\langle\alpha^\vee,\mu+a\rho\rangle}}{1 - tq^{\langle\alpha^\vee,\mu+a\rho\rangle}}. \tag{A.3}$$

We show that the sum on the left-hand side of the above equation can be replaced by the sum on $\nu \in W(\mu)$. Consider the isotropy group $W_\mu = \{w \in W | w(\mu) = \mu\}$ for the partition $\mu \in P_+$. Since an element $w \in W_\mu \setminus \{1\}$ can be written by a product of simple reflections fixing μ, $\{s_i | i \in J \subset I\}$, there exists at least one simple root $\alpha_i \in \Pi$ associated with the reflection s_i in the set R_w. Hence, for $w \in W_\mu \setminus \{1\}$, we have

$$\begin{aligned}
\prod_{\alpha^\vee\in R_w^\vee} t(1 - t^{-1} q^{\langle\alpha^\vee,\mu+a\rho\rangle}) &= t(1 - t^{-1} q^{\langle\alpha_i^\vee,a\rho\rangle}) \prod_{\alpha\in R_w\setminus\{\alpha_i\}} t(1 - t^{-1} q^{\langle\alpha^\vee,\mu+a\rho\rangle}) \\
&= t(1 - t^{-1} t) \prod_{\alpha\in R_w\setminus\{\alpha_i\}} t(1 - t^{-1} q^{\langle\alpha^\vee,\mu+a\rho\rangle}) = 0.
\end{aligned}$$

Define $W^\mu := \{w \in W | \ell(ws_i) > \ell(w) \text{ forall } i \in J\}$. For $w \in W$, there is a unique $u \in W^\mu$ and a unique $v \in W_\mu$ such that $w = uv$. The formula $R_w = R_v \cup v^{-1} R_u$ shows that, if $v \neq 1$, the product on $\alpha \in R_w$ in (A.3) vanishes. Thus we obtain the above lemma since the sum on $w \in W$ on the left-hand side of (A.3) can be replaced by that on $u \in W^\mu$ which is equivalent to that on $\nu \in W(\mu)$. □

In the formal limit $q \to 1$ under the restriction $t = q^a$, we have the relation in lemma 5.2. The formula in the anti-symmetric case is proved in a similar way.

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